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Hecke algebraic approach to the reflection equation for spin chains

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Abstract
We use the structural similarity of certain Coxeter Artin systems to the Yang–Baxter and reflection equations to convert representations of these systems into new solutions of the reflection equation. We construct certain Bethe ansatz states for these solutions, using a parametrization suggested by abstract representation theory.

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1. Introduction and review

There has been much interest recently in the role of boundaries in integrable systems, both from the point of view of critical phenomena (see for example [1] and references therein), and integrability [2]. There has also been considerable progress in constructing representations of affine Hecke algebras [3, 4] with global (i.e. quasi-thermodynamic) limits [5, 6]. In this paper we apply this algebraic technology to the boundary R-matrix problem, in a way analogous to the use by many authors of the ordinary Hecke algebra in solving the Yang–Baxter equations (see [7, 8] for reviews).

We start by briefly reviewing the standard R-matrix formulation of the Yang–Baxter equation (YBE) in the context of spin chains, and the Hecke/Temperley–Lieb algebraic variant of this formulation. We then generalize to K-matrices and boundary YBE—i.e. to the reflection equation (RE) [9, 10]. In section 2 we discuss the algebraic structures with roles *analogous* to the ordinary Hecke and Temperley–Lieb algebras in the boundary case, and give a number of constructions for representations of such algebras, which representations provide candidates for solutions to RE. In section 3 we show that the resultant 'blob algebra' b_n indeed provides new (and well parametrized) solutions to RE. Finally, we look at the Bethe ansatz for some intriguing 'spin-chain-like' representations of this algebra.

The parallels with the ordinary closed boundary $U_q sl_2$-invariant spin-chain case are strong, but the symmetry algebra is *not* always $U_q sl_2$. This raises some very interesting questions for further study. The representation theory of b_n has parallels with that of the Virasoro

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algebras arising in conformal field theory, and the Bethe ansatz may provide a mechanism for investigating this (cf [8, 11–13]).

Fix integers $N > 0$ and $n \gg 0$, and let V be a complex N-space. Write $V^n = \otimes_{i=1}^n V$. For $N = 2$ the Pauli σ-matrices, and indeed $U_q sl_2$, act naturally on V, and V^n is the underlying space of the n-site XXZ model. Define $H_n^N(q) = \mathrm{End}_{U_q sl_N}(V^n)$. The ordinary Temperley–Lieb algebra $T_n(q)$ (see later) is isomorphic to $H_n^2(q)$.

1.1. R-matrices

Define $\mathcal{P}$ to act on $V \otimes V$ by $\mathcal{P}x \otimes y = y \otimes x$. If A is any matrix acting on $V^m = \otimes_{i=1}^m V$, and $i_1, \ldots, i_m \leqslant n$ distinct natural numbers, then (in 'R-index notation') $A_{i_1 \ldots i_m}$ acts on V^n by embedding the A action onto the $i_1^{\mathrm{th}} \cdots i_m^{\mathrm{th}}$ factors V. For example, $\mathcal{P}_{12} = \mathcal{P}_{21}$ and $\mathcal{P}_{12}\mathcal{P}_{13}\mathcal{P}_{12} = \mathcal{P}_{23}$. Dually, if T is a matrix acting on $V \otimes V^n$ (with factors indexed from $0, 1, \ldots, n$) then T_0 is T regarded as an $N^n \times N^n$-matrix-valued $N \times N$-matrix in the obvious way. Generalizing this (for a moment) so that T_i is T expanded with respect to the ith factor then $\mathrm{Tr}_i(T) = \mathrm{Tr}(T_i)$, the trace (we may also write this as $\mathrm{Tr}_i(T_i)$); and $T^{t_i} = (T_i)^t$, the transpose.

An (adjoint) $R(\lambda)$-*matrix* is a matrix acting on V^2 which solves the Yang–Baxter equation in the (R-index) form [14]

$$R_{12}(\lambda - \lambda')R_{13}(\lambda)R_{23}(\lambda') = R_{23}(\lambda')R_{13}(\lambda)R_{12}(\lambda - \lambda'). \tag{1}$$

We also require *unitarity*:

$$R_{12}(\lambda)R_{21}(-\lambda) \propto 1 \tag{2}$$

(note that, $R_{21}(\lambda) = \mathcal{P}_{12}R_{21}(\lambda)\mathcal{P}_{12}$); $R_{21}(\lambda) = R_{21}(\lambda)^{t_1 t_2}$; and [15] that there exist $M = M^t$ and ρ such that

$$R_{12}(\lambda)^{t_1} M_1 R_{12}(-\lambda - 2\rho)^{t_2} M_1^{-1} \propto 1 \tag{3}$$

$$[M_1 M_2, R_{12}(\lambda)] = 0. \tag{4}$$

Given such an $R(\lambda)$-matrix, introduce *monodromy matrix* [16, 17]

$$T(\lambda) = R_{0n}(\lambda) \cdots R_{01}(\lambda). \tag{5}$$

Note that this acts on $V \otimes V^n = V_0 \otimes V_1 \otimes V_2 \cdots V_n$. Spaces V_i ($i > 0$) are called 'quantum'; space V_0 is called 'lateral' or 'auxiliary'. One often makes manifest just the lateral space subscript: $T(\lambda) = T_0(\lambda)$. The YBE implies

$$R_{00'}(\lambda - \lambda')T_0(\lambda)T_{0'}(\lambda') = T_{0'}(\lambda')T_0(\lambda)R_{00'}(\lambda - \lambda'). \tag{6}$$

There is a convenient pictorial realization of the YBE and of equation (6) in, for example, [18].

The *closed chain transfer matrix* is

$$t(\lambda) = \mathrm{Tr}_0 T_0(\lambda). \tag{7}$$

By virtue of (6) and the existence of the inverse of $R(\lambda)$ this obeys

$$[t(\lambda), t(\lambda')] = 0. \tag{8}$$

For example, with $N = 2$ the XXZ model with anisotropy parameter $\mu \geqslant 0$ has [18]

$$R(\lambda) = \begin{pmatrix} a(\lambda) & & & \\ & b(\lambda) & c_+(\lambda) & \\ & c_-(\lambda) & b(\lambda) & \\ & & & a(\lambda) \end{pmatrix} \tag{9}$$

where

$$
\begin{aligned}
a(\lambda) &= \sinh(\mu(\lambda + \mathrm{i})) \\
b(\lambda) &= \sinh(\mu\lambda) \\
c_{\pm}(\lambda) &= \sinh(\mathrm{i}\mu)\, \mathrm{e}^{\pm\mu\lambda}
\end{aligned}
\tag{10}
$$

(also known as the $A_1^{(1)}$ case, by an association with the $A_1^{(1)}$ affine Lie algebra). This R-matrix obeys (3) and (4) with [19, 20]

$$
M_{jk} = \delta_{jk}\, \mathrm{e}^{\mathrm{i}\mu(3-2j)} \qquad \rho = i. \tag{11}
$$

1.2. R-matrices and the TL algebraic method

Given an R-matrix, set

$$
\check{R}_{ii+1}(\lambda) = \mathcal{P}_{ii+1} R_{ii+1}(\lambda) = R_{i+1i}(\lambda)\mathcal{P}_{ii+1}. \tag{12}
$$

Premultiplying (1) by $\mathcal{P}_{23}\mathcal{P}_{12}\mathcal{P}_{23}$ we get

$$
\check{R}_{12}(\lambda - \lambda')\check{R}_{23}(\lambda)\check{R}_{12}(\lambda') = \check{R}_{23}(\lambda')\check{R}_{12}(\lambda)\check{R}_{23}(\lambda - \lambda'). \tag{13}
$$

What is deep about (1) is the construction of commuting transfer matrices, and this is not restricted to, and may be abstracted away from, the V^n setting. One introduces abstract operators $\check{R}_i(\lambda)$ (not in R-index notation) obeying

$$
\check{R}_i(\lambda - \lambda')\check{R}_{i+1}(\lambda)\check{R}_i(\lambda') = \check{R}_{i+1}(\lambda')\check{R}_i(\lambda)\check{R}_{i+1}(\lambda - \lambda') \tag{14}
$$

and

$$
\check{R}_i(\lambda)\check{R}_j(\lambda') = \check{R}_j(\lambda')\check{R}_i(\lambda) \qquad i - j > 1. \tag{15}
$$

This is called the *Hecke algebraic form* of the YBE. It will be evident that every R-matrix gives a solution to these equations via the substitution $\check{R}_i(\lambda) \mapsto \check{R}_{ii+1}(\lambda)$.

The abstract Temperley–Lieb algebra $T_n(q)$ is generated by the unit element and elements $U_1, \ldots, U_{n-1}$ satisfying the following relations [21, 22]:

$$
\begin{aligned}
&U_i U_i = -(q + q^{-1})U_i \qquad q = \mathrm{e}^{\mathrm{i}\mu} \\
&U_i U_{i\pm1} U_i = U_i \\
&[U_i, U_j] = 0 \qquad i - j > 1.
\end{aligned}
\tag{16}
$$

Let $N = 2$, and V^n the corresponding tensor space with action of $U_q sl_2$ [7, 23]. Set

$$
\begin{aligned}
\mathcal{R}(U_i) = \mathcal{R}_q(U_i) &= \sigma_i^+\sigma_{i+1}^- + \sigma_i^-\sigma_{i+1}^+ + \frac{q + q^{-1}}{1}\left(\sigma_i^z\sigma_{i+1}^z - \frac{1}{4}\right) + \frac{q - q^{-1}}{2}\left(-\sigma_i^z + \sigma_{i+1}^z\right) \\
&= 1 \otimes \cdots \otimes \mathcal{U} \otimes \cdots \otimes 1
\end{aligned}
\tag{17}
$$

where

$$
\mathcal{U} = \begin{pmatrix} 0 & & & \\ & -\mathrm{e}^{\mathrm{i}\mu} & 1 & \\ & 1 & -\mathrm{e}^{-\mathrm{i}\mu} & \\ & & & 0 \end{pmatrix} \tag{18}
$$

(i.e. the nontrivial part is a 4×4 matrix acting on $V_i \otimes V_{i+1}$, so $\mathcal{R}(U_i) = \mathcal{U}_{ii+1}$ in R-index notation).

Proposition 1 [24]**.** *The matrices $\mathcal{R}(U_i)$ define a representation of $T_n(q)$ which is (i) faithful; and (ii) commutes with the action of $U_q sl_2$ on V^n.*

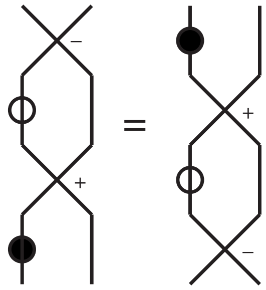

Figure 1. Pictorial realization of the RE.

For the XXZ R-matrix of equation (9) we find

$$\check{R}_{ii+1}(\lambda) = \sinh(\mu(\lambda + \mathrm{i}))1 + \sinh(\mu\lambda)\mathcal{R}(U_i). \tag{19}$$

Thus $\mathcal{R}$ gives a solution to (13) and hence to (1). Since $\mathcal{R}$ is faithful, any representation of $T_n(q)$ would give a solution to (14). We say $T_n(q)$ gives a *meta-solution*.

1.3. K-matrices

Given an R-matrix, a $K(\lambda)$-*matrix* acts on V and obeys the reflection equation [9]:

$$R_{12}(\lambda_1 - \lambda_2)K_1(\lambda_1)R_{21}(\lambda_1 + \lambda_2)K_2(\lambda_2) = K_2(\lambda_2)R_{12}(\lambda_1 + \lambda_2)K_1(\lambda_1)R_{21}(\lambda_1 - \lambda_2). \tag{20}$$

We require $K(0) = 1$ and $K(\lambda)K(-\lambda) \propto 1$. Using this one may construct commuting *open* boundary transfer matrices and solve the corresponding Bethe ansatz equations [10].

A suitable transfer matrix $t(\lambda)$ for an open chain of n spins is [10, 23, 25]

$$t(\lambda) = \mathrm{Tr}_0 M_0 K_0^+(-\lambda - \rho)^t T_0(\lambda) K_0^-(\lambda)\hat{T}_0(\lambda) \tag{21}$$

where

$$\hat{T}_0(\lambda) = R_{10}(\lambda)\cdots R_{n0}(\lambda) \tag{22}$$

$K^-(\lambda) = K(\lambda)$ where the $K(\lambda)$ is a solution of the reflection equation, and K^+ satisfies an equation similar to (20) [26] (we can and will set $K^+ = 1$ without significant loss of generality).

Following Sklyanin [10] define

$$\mathcal{T}(\lambda) = T_0(\lambda)K_0^-(\lambda)\hat{T}_0(\lambda) \tag{23}$$

which satisfies

$$R_{12}(\lambda_1 - \lambda_2)\mathcal{T}_1(\lambda_1)R_{21}(\lambda_1 + \lambda_2)\mathcal{T}_2(\lambda_2) = \mathcal{T}_2(\lambda_2)R_{12}(\lambda_1 + \lambda_2)\mathcal{T}_1(\lambda_1)R_{21}(\lambda_1 - \lambda_2). \tag{24}$$

We may again use a pictorial representation to see this. Following the realization in [18] (or, specifically, [27, figure 1]), the picture for the reflection equation (20) is as in figure 1. In this realization the Sklyanin operator appears as in figure 2. The identity (24) follows in the manner of figure 3.

The transfer matrix also obeys

$$[t(\lambda), t(\lambda')] = 0. \tag{25}$$

Consider the XXZ$\big/A_1^{(1)}$ R-matrix as before. For $K = 1$,

$$[t(\lambda), g] = 0 \tag{26}$$

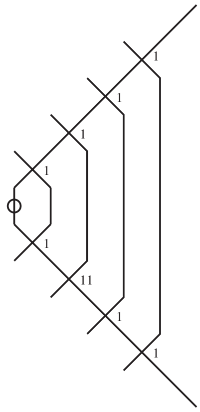

Figure 2. The Sklyanin operator $\mathcal{T}(\lambda)$.

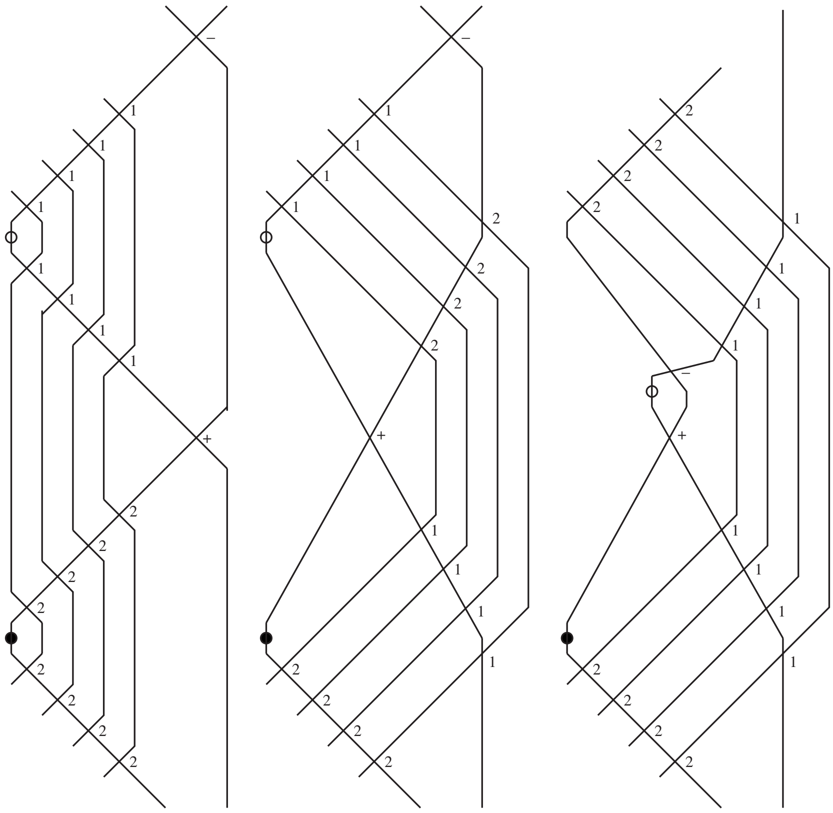

Figure 3. First steps in verification of commutation. Step 1 is an application of YBE as in [18], [27, figure 1]. Step 2 is similar. At this point the left-hand side of RE has appeared in the picture. One applies RE to it and then completes the manipulation by further applications of YBE.

where g is the usual $U_q sl_2$ action [10, 28–30]. The symmetry for the general diagonal K is more complicated (see e.g. [26]).

In the Temperley–Lieb notation the RE is

$$\check{R}_1(\lambda_1-\lambda_2)\check{K}(\lambda_1)\check{R}_1(\lambda_1+\lambda_2)\check{K}(\lambda_2)=\check{K}(\lambda_2)\check{R}_1(\lambda_1+\lambda_2)\check{K}(\lambda_1)\check{R}_1(\lambda_1-\lambda_2). \tag{27}$$

As we will now see, this makes it natural to seek solutions among the affine generalizations of $T_n(q)$.

2. (Affine) braids and Hecke algebras

Recall that a *Coxeter graph* G is any finite undirected graph without loops (almost everybody's attention is habitually restricted to the subset of graphs of *positive type* [31, section 2.3]). For given G let $m(s,s')$ denote the number of edges between vertices s and s'. The *Coxeter system* of G is a pair (W,S) consisting of a group W and a set S of generators of W labelled by the vertices of G, with relations of the form

$$g_s g_{s'} g_s g_{s'} \ldots = g_{s'} g_s g_{s'} g_s \ldots \tag{28}$$

where the number of factors on each side is $m(s,s')+2$; and

$$g_s^{-1}=g_s. \tag{29}$$

If we relax the set of relations in (29) (and add as generators the inverse of each $g_s\in S$) we get a *Coxeter Artin system*, and $W=\mathcal{A}_G$ is an *Artin group* [32]. For example, let $\mathcal{B}_n$ denote the ordinary Artin braid group, the group of composition of finite braidings of n strings running from the northern to the southern edge of a rectangular frame. Then $\mathcal{A}_{A_{n-1}}\cong\mathcal{B}_n$.

In the case $G=B_n$ the (non-commuting) relations may be written as

$$g_0g_1g_0g_1=g_1g_0g_1g_0 \tag{30}$$

$$g_ig_{i+1}g_i=g_{i+1}g_ig_{i+1} \qquad n-1>i\geqslant 1. \tag{31}$$

And here is the point *of this excursion*: we will use the structural similarity of these relations to the reflection equation (RE) [9, 10] and Yang–Baxter equation (YBE) [22] to develop various realizations of $\mathcal{A}_{B_n}$ into candidates for solutions to these equations. There are two parts to this task. Finding quotients of the braid group in which (30) and (31) may be deformed to solve RE and YBE, respectively (see section 2.3); and then finding realizations of these quotients suitable for Bethe ansatz formulation. Our approach to the latter problem is to borrow from what works in the ordinary case [22]. Thus we have to make *contact* with the ordinary case. We do this next.

2.1. Boundaries, cylinder braids and $\mathcal{A}_{B_n}$

Let $\mathcal{B}_n^{\mathrm{o}}$ denote the Artin braid group on the cylinder (or annulus—the correspondence between the cylinder and annulus versions is trivial, cf [33, 34], and we will use them interchangeably). Figure 4 shows some elements of $\mathcal{B}_3^{\mathrm{o}}$ (together with an assertion, to be verified later, of their preimages in $\mathcal{A}_{B_3}$ under a certain group homomorphism). Figure 5 illustrates the composition in the cylinder braid group, and the Reidemeister move [35, III section 1] of type 2 in this context ([35] provides a summary of and link to Reidemeister's original works). There is an obvious inclusion $\iota:\mathcal{B}_n\hookrightarrow\mathcal{B}_n^{\mathrm{o}}$ got by identifying the right and left edges of the frame. There is an obvious surjective homomorphism $\sigma:\mathcal{B}_n^{\mathrm{o}}\to\mathcal{B}_n$ got by arranging for all the string endpoints to be gathered on one side of the cylinder and then squashing the cylinder flat with this side on top[3]. Note that $\tau=g_{n-1}\ldots g_2g_1c_0$ is a useful *twist* element.

[3] Most of the groups we consider here contain $\mathcal{B}_m$ as a subgroup at least for some m. For example, if A_{m-1} is a full subgraph of G then $\mathcal{A}_G\supset\mathcal{B}_m$. Where it is unambiguous to do so we will refer to the elements which lie in this subgroup by their $\mathcal{B}_m$ names (thus g_1 and so on).

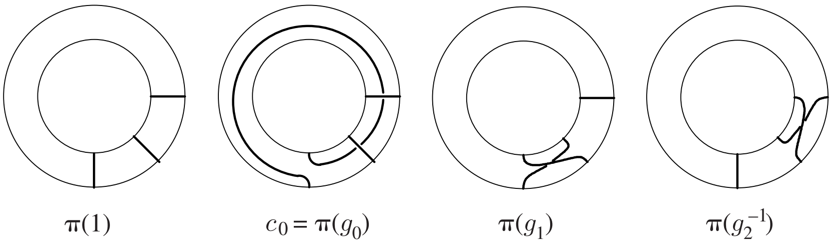

Figure 4. Elements of the 3-string braid group on the cylinder.

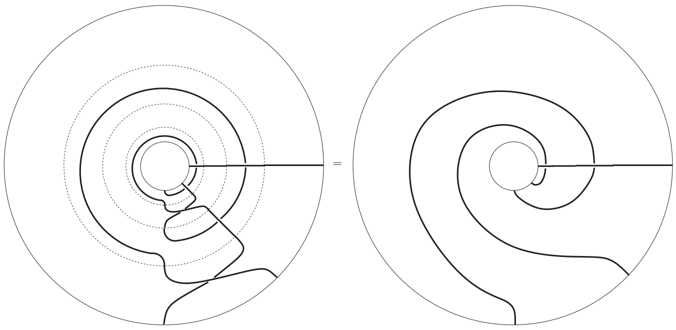

Figure 5. The composition $c_0 g_1 c_0 g_1$—a demonstration of the image of relation (30) in a cylinder braid group.

Proposition 2 [27]. *Each of the sets* $S = \{c_0^{\pm 1}, g_1^{\pm 1}, g_2^{\pm 1}, \dots\}$ *and* $S' = \{\tau^{\pm 1}, g_1^{\pm 1}, g_2^{\pm 1}, \dots\}$ *generates* $\mathcal{B}_n^{\mathrm{o}}$.

The interplay between B-type and periodic algebraic systems and boundary conditions for YBE (cf (30), (31)) is neatly summed up by the following.

Proposition 3. *There is a group homomorphism*

$$\pi : \mathcal{A}_{B_{n+1}} \longrightarrow \mathcal{B}_n^{\mathrm{o}}$$

in which the images of the set $\{g_0, g_1, g_2, \dots\}$ *of generators are (the generators) as indicated in figure 4.*

Figure 5 verifies the special relation (30) in this realization, in as much as it is manifest from the RHS that the (outer) factor of $\pi(g_1)$ commutes with the rest of the diagram. Note from proposition 2 that π is surjective. (And see [36, 37].)

It will be evident that there is a homomorphism from $\mathcal{A}_{\hat{A}_{n+1}}$ (with generator $\hat{g}_{n+1}$, say, where vertex $n+1$ is adjacent to both 1 and n in $\hat{A}_{n+1}$) into $\mathcal{B}_{n+1}^{\mathrm{o}}$. This may be given in our $n = 2$ example as $\hat{g}_3 \mapsto \tau g_1 \tau^{-1}$.

2.2. On maps into the ordinary braid group

Recall that the pure braid group $\mathcal{B}'_n$ is normal in $\mathcal{B}_n$, and that the quotient defines a surjection $P : \mathcal{B}_n \to S_n$ onto the symmetric group. For p a partition of $\{1, 2, \dots, n\}$, the subset of permutations which fixes p forms a subgroup, called the *Young subgroup* S_p of S_n. We may extend this to define a subgroup $\mathcal{B}_p$ of $\mathcal{B}_n$ which fixes p in the sense that braid b fixes p if $P(b)$

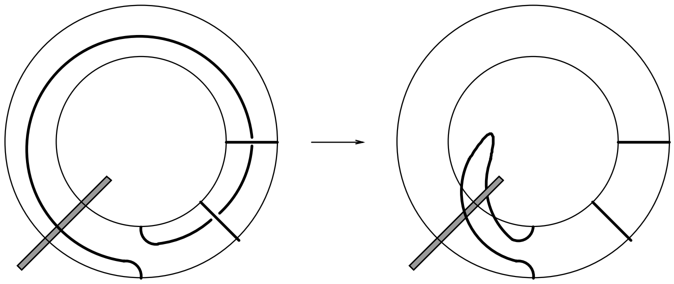

Figure 6. Inserting strings into the cylinder.

does. For each p_i, a part of p, there is a natural 'restricting' map from $\mathcal{B}_p$ onto $\mathcal{B}_{|p_i|}$ which simply ignores all strings not in p_i.

For $m = 1, 2, \ldots$, let $\mathcal{B}^m_{n+m}$ denote the subgroup of $\mathcal{B}_{n+m}$ in which the first m strings are pure. Let J_n denote the subgroup of $\mathcal{B}_{2n}$ consisting of braids which are invariant under rotation about an axis passing north to south, starting halfway between the nth and $(n+1)$th northern endpoints (as exemplified in figure 7(a)).

Next we establish maps between $\mathcal{A}_{B_n}$ and $\mathcal{B}^{\text{o}}_n$ and these subgroups of $\mathcal{B}_n$ which enable us to port information between them. This is useful as each brings a particular utility to the problem of their analysis ($\mathcal{B}^{\text{o}}_n$ has nice diagrams, and periodicity; $\mathcal{B}^1_{n+1}$ forms a tower of subalgebras on varying n, and has representations by restriction from $\mathcal{B}_n$; and $\mathcal{A}_{B_n}$ has direct structural similarity with RE and the blob algebra (see later)).

There is a mapping

$$\sigma_l : \mathcal{B}^{\text{o}}_n \to \mathcal{B}^l_{n+l}$$

like σ, but which keeps track of which strings actually went round the back of the cylinder (i.e., it is injective). Before squashing the cylinder completely flat we slide an extra row of l mutually non-crossing strings into the hole, pushing them over so that they lie at, say, the left-hand end of the row of strings in the squashed cylinder (see figure 6). For example $\sigma_1(c_0) = g_1^2, \sigma_1(g_i) = g_{i+1} (i > 0)$. To see that this map is injective note that the strings which went round the back now go round the extra strings in the appropriate sense (so the manoeuvre is reversible). The image of this map is a nonempty subgroup of $\mathcal{B}^l_{n+l}$ which restricts, on the first l strings, to the trivial group. Note, then, that σ_1 is an *isomorphism*. We will again use these two realizations interchangeably where no confusion arises (cf [38–40]). Indeed, for mapping the braid groups themselves the generalization to $l > 1$ is effectively spurious. We include it because we will later want to study the maps induced by σ_l on quotient algebras, and these maps do depend on l (and even on variations like attaching an idempotent to the first l strings [6]).

There is an injective homomorphism $\gamma : \mathcal{B}_n \to J_n$ given by

$$\gamma : g_i \mapsto g_{n-i} g_{n+i}. \tag{32}$$

This extends to a homomorphism $\gamma : \mathcal{A}_{B_{n+1}} \to J_n$ by

$$\gamma : g_0 \mapsto g_n$$

(see figure 7(a)). Physicists will recognize an analogy in this with the *method of images*. There is a similar extension of the cabling map [6].

Without the extension, the map γ is essentially the group comultiplication $\Delta : \mathcal{B}_n \to \mathcal{B}_n \times \mathcal{B}_n$ embedded, Young subgroup style, in $\mathcal{B}_{2n}$. Recall that this equips the group algebra with the property of bialgebra (indeed Hopf algebra); and implies that the category of left

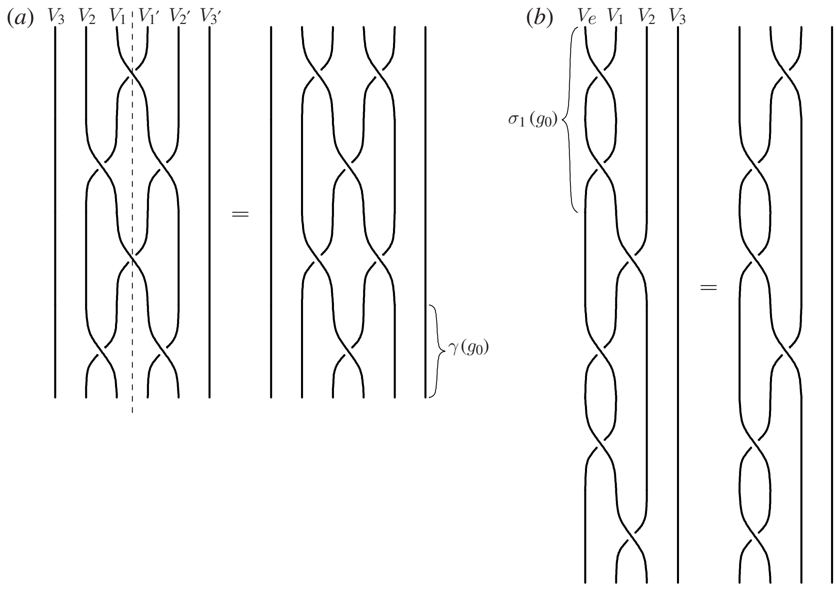

Figure 7. (*a*) Element $\gamma(g_0g_1g_0g_1)$ of $J_3 \subset \mathcal{B}_6$, showing (dashed) symmetry axis. (*b*) Element $\sigma_1(g_0g_1g_0g_1)$ of $\mathcal{B}^1_{3+1} \subset \mathcal{B}_4$.

modules is closed under tensor products (see [41] for example). There is a generalization of this (see [6, 42, section A(iii)]) which enables us to close the sum over $q \in \mathbb{C}$ of categories of left $T_n(q)$-modules under tensor products. It is possible to extend the representation obtained by tensoring two copies of the ordinary spin-chain representation (as in equation (17)) to a representation of $\mathcal{B}^{\mathrm{o}}_n$ [6]. We will recall the precise construction in section 5. This is in particular a faithful two-parameter representation of the blob algebra b_n [40], which is a quotient of $\mathcal{B}^{\mathrm{o}}_n$ which explicitly solves RE—see section 3. As such this representation is arguably the most interesting candidate for studying spin chains with boundary currently available. There are other possibilities, however, as we now summarize.

2.3. *Quotient algebras and representations*

The above discussion gives us a number of recipes for constructing representations of cylinder algebras from those of $\mathbb{C}\mathcal{B}_n$. Many $\mathbb{C}\mathcal{B}_n$ representations may be used to construct exactly solvable models, so applying the recipes to these should provide good candidates for ESMs with more general boundary conditions. Unfortunately, these representations have important properties which are not necessarily preserved by passage to the cylinder. When $\mathbb{C}\mathcal{B}_n$ is used to solve the YBE it is never, physically, a faithful representation which appears (and the vanishing of the annihilator is *used* in the solution). Indeed, on physical representations each g_s has a finite spectrum.

If each g_s has spectrum of order 2 then we are in the realm of *generic algebras* [31] (natural generalizations of the corresponding Coxeter systems (W, S) in which, of course, $g_s^2 = 1$ for all $s \in S$). In a generic algebra g_s and g_t have the same spectrum if s, t conjugate in W. Thus in the A_n case each g_s has the same spectrum—we write

$$(g_i - q)(q_i + q^{-1}) = 0 \tag{33}$$

whereupon we have the ordinary Hecke algebra $H_n(q)$ [43]. Although $H_n(q)$ is a relatively tiny vestige of $\mathbb{C}\mathcal{B}_n$, even this algebra is never faithfully represented in physical representations (and no global limit of the whole of $H_n(q)$ is known). A natural example of a quotient of $\mathbb{C}\mathcal{B}_n$ which *does* have a global limit is the Temperley–Lieb algebra [7].

We may assume that a similar situation pertains in the 'affine' case. Applying (33) to $\mathbb{C}\mathcal{A}_{B_n}$ we get an *affine Hecke algebra* [44], again too large to be physical. A number of potentially suitable quotients are discussed in [5, 6]. The $N = 2$ case (an affine equivalent of Temperley–Lieb) is the aforementioned *blob algebra*. It has been examined in some detail from the ordinary representation theory viewpoint [40]. On the other hand, while $\mathbb{C}\mathcal{B}_n$ and its quotients all have a natural inclusion via $A_n \subset A_{n+1}$, and a number of physically useful representations are known, embedding cylinder algebras in towers is somewhat harder. The preceding discussion provides solutions to this problem by building cylinder algebras out of ordinary ones. The price paid is that while these constructions work at the level of braids, they do not in general factor through the quotients which we are obliged to restrict to physically. The remainder of this paper is concerned with finding cases which *do* factor, and using these to solve the reflection equation. We typically have some variant of the following picture:

$$\begin{tikzcd}
\mathbb{C}\mathcal{B}_n^c \arrow[r, "\sigma."] \arrow[d, "\Psi^2"] \arrow[drr] & \mathbb{C}\mathcal{B}_{n+l}^l \arrow[r, hook] & \mathbb{C}\mathcal{B}_{n+l} \arrow[d] \\
b_n \arrow[rr, dotted, "\Theta?"] & & \mathrm{End}(V^{n+l}).
\end{tikzcd}$$

Here $\sigma_.$ represents any of the maps constructed in section 2.2; the diagonal map is defined by the commutativity of the upper triangle; Ψ^2 is the quotient map to the blob algebra (see section 3) or some other suitable quotient; and Θ is the representation of b_n we get *if* the diagonal map factors through b_n.

Solutions which do not start with XXZ, or do not end up in the blob quotient, raise rather different problems, and will be examined in a separate paper.

3. The abstract blob algebra solution

In this section we look for solutions to the reflection equation based on the special representations of B-braids discussed above. We show that the abstract blob algebra provides a meta-solution in the same sense as the Temperley–Lieb algebra does for the ordinary YBE.

The blob algebra $b_n = b_n(q, m)$ may be defined by generators $U_1, U_2, \ldots, U_{n-1}$ and e, and relations:

$$U_i U_i = \delta U_i \tag{34}$$

$$U_i U_{i\pm1} U_i = U_i \tag{35}$$

$$[U_i, U_j] = 0 \qquad |i - j| \neq 1 \tag{36}$$

(so far we have the ordinary Temperley–Lieb algebra with $-\delta = q + q^{-1}$)

$$ee = \delta_e e \tag{37}$$

$$\begin{aligned} U_1 e U_1 &= \kappa U_1 \\ [U_i, e] &= 0 \qquad i \neq 1. \end{aligned} \tag{38}$$

Note that we are free to renormalize e, changing only δ_e and κ (by the same factor), thus from δ, δ_e, κ there are really only two relevant parameters. It will be natural later on to reparametrize so that they are related (they only depend on q and m), but it will be convenient to treat them separately for the moment, and leave m hidden.

Assuming for the moment that we have some viable representation of this algebra we may proceed as follows. Setting

$$\begin{aligned} R_1(\theta_1 \pm \theta_2) &= a_\pm 1 + b_\pm U_1 \\ K(\theta_i) &= x_i 1 + y_i e \end{aligned} \tag{39}$$

the reflection equation

$$R_1(\theta_1 - \theta_2)K(\theta_1)R_1(\theta_1 + \theta_2)K(\theta_2) = K(\theta_2)R_1(\theta_1 + \theta_2)K(\theta_1)R_1(\theta_1 - \theta_2)$$

becomes

$$\begin{aligned} &(a_- 1 + b_- U_1)(x_1 1 + y_1 e)(a_+ 1 + b_+ U_1)(x_2 1 + y_2 e) \\ &\qquad = (x_2 1 + y_2 e)(a_+ 1 + b_+ U_1)(x_1 1 + y_1 e)(a_- 1 + b_- U_1) \end{aligned}$$

and hence

$$\begin{aligned} &(a_- x_1 1 + a_- y_1 e + b_- x_1 U_1 + b_- y_1 U_1 e)(a_+ x_2 1 + a_+ y_2 e + b_+ x_2 U_1 + b_+ y_2 U_1 e) \\ &\qquad = (a_+ x_2 1 + a_+ y_2 e + b_+ x_2 U_1 + b_+ y_2 e U_1)(a_- x_1 1 + a_- y_1 e + b_- x_1 U_1 + b_- y_1 e U_1) \end{aligned}$$

and hence

$$\begin{array}{rcr} a_- a_+ x_1 x_2 1 & & a_- a_+ x_1 x_2 1 \\ + a_- a_+ (x_1 y_2 + x_2 y_1 + \delta_e y_1 y_2) e & & + a_- a_+ (x_1 y_2 + x_2 y_1 + \delta_e y_1 y_2) e \\ + (a_- b_+ + a_+ b_- + \delta b_+ b_-) x_1 x_2 U_1 & & + (a_- b_+ + a_+ b_- + \delta b_+ b_-) x_1 x_2 U_1 \\ + ((a_- b_+ + a_+ b_- + \delta b_+ b_-) x_1 y_2 + & & \\ a_+ b_- (x_2 y_1 + \delta_e y_1 y_2)) U_1 e & & + a_- b_+ x_2 y_1 U_1 e \\ & = & + ((a_- b_+ + a_+ b_- + \delta b_+ b_-) x_1 y_2 + \\ a_- b_+ x_2 y_1 e U_1 & & + a_+ b_- (x_2 y_1 + \delta_e y_1 y_2)) e U_1 \\ + a_- b_+ y_1 y_2 e U_1 e & & + a_- b_+ y_1 y_2 e U_1 e \\ + b_- b_+ y_1 x_2 U_1 e U_1 & & + b_- b_+ y_1 x_2 U_1 e U_1 \\ + b_- b_+ y_1 y_2 U_1 e U_1 e & & + b_- b_+ y_1 y_2 e U_1 e U_1 \end{array}$$

Now applying relation (38) this becomes

$$((a_- b_+ + a_+ b_- + \delta b_+ b_-) x_1 y_2 + a_+ b_- (x_2 y_1 + \delta_e y_1 y_2) - c_- b_+ x_2 y_1 + \kappa b_- b_+ y_1 y_2)[e, U_1] = 0.$$

Dividing by $y_1 y_2$ and putting $k_i = \frac{x_i}{y_i}$ we have

$$(a_- b_+ + a_+ b_- + \delta b_- b_+) k_1 + (-a_- b_+ + a_+ b_-) k_2 + (\delta_e a_+ b_- + \kappa b_+ b_-) = 0$$

so

$$A_2 k_2 = A_1 k_1 + B$$

where $A_1 = (a_- b_+ + a_+ b_- + \delta b_- b_+)$, $B = (\delta_e a_+ b_- + \kappa b_+ b_-)$ and $A_2 = (a_- b_+ - a_+ b_-)$. Since k_i can depend only on θ_i this equation is required to separate for a solution.

Recalling that $q = \mathrm{e}^{\mu \mathrm{i}}$, then $a_\pm = \mathrm{sh}(\mu(\theta_1 \pm \theta_2 + \mathrm{i}))$, $b_\pm = \mathrm{sh}(\mu(\theta_1 \pm \theta_2))$ are inherited from the global YB solution. Thus

$$\begin{aligned} A_1 &= \mathrm{sh}(\mu(\theta_1 - \theta_2 + \mathrm{i}))\,\mathrm{sh}(\mu(\theta_1 + \theta_2)) + \mathrm{sh}(\mu(\theta_1 + \theta_2 + \mathrm{i}))\,\mathrm{sh}(\mu(\theta_1 - \theta_2)) \\ &\qquad - 2\mathrm{sh}(\mu(\theta_1 + \theta_2))\,\mathrm{sh}(\mu(\theta_1 - \theta_2))\,\mathrm{ch}(\mu \mathrm{i}) \\ &= \mathrm{sh}(\mu 2\theta_1)\,\mathrm{sh}(\mu \mathrm{i}) \\ A_2 &= \mathrm{sh}(\mu 2\theta_2)\,\mathrm{sh}(\mu \mathrm{i}) \\ B &= \delta_e\,\mathrm{sh}(\mu(\theta_1 - \theta_2))\,\mathrm{sh}(\mu(\theta_1 + \theta_2 + \mathrm{i})) + \kappa\,\mathrm{sh}(\mu(\theta_1 - \theta_2))\,\mathrm{sh}(\mu(\theta_1 + \theta_2)) \\ &= \tfrac{1}{2}(\delta_e(\mathrm{ch}(\mu(2\theta_1 + \mathrm{i})) - \mathrm{ch}(\mu(2\theta_2 + \mathrm{i}))) + \kappa(\mathrm{ch}(\mu(2\theta_1)) - \mathrm{ch}(\mu(2\theta_2)))) \end{aligned}$$

so we *may* separate to obtain

$$\mathrm{sh}(\mu 2\theta_j)\,\mathrm{sh}(\mu i)k_j = -\tfrac{1}{2}(\delta_e\,\mathrm{ch}(\mu(2\theta_j + \mathrm{i})) + \kappa\,\mathrm{ch}(\mu(2\theta_j)) + \mathrm{ch}(\mu 2\mathrm{i}\zeta))$$

where ζ is the (arbitrary) constant of separation. At this point we have established a *solution to* RE (or rather a meta-solution which produces a solution for each representation of b_n). The blob algebra is a quotient of a special case of the algebras shown to solve RE in [5, 45], which guarantees that it gives a solution in principle. However, the precise form of b_n leads to a significant and crucial simplification in parametrization, cf the general case. This is even more striking when we apply the parametrization known from representation theory, as follows.

Recall $[m] = \frac{\mathrm{sh}(m\mu \mathrm{i})}{\mathrm{sh}(\mu \mathrm{i})}$. In the abstract form a natural parametrization of the two-parameter algebra b_n is $\delta = -[2]$, $\delta_e = -[m]$, $\kappa = [m-1]$ (the two parameters are q and m), and hence

$$\begin{aligned}\mathrm{sh}(\mu 2\theta_j)\,\mathrm{sh}(\mu \mathrm{i})k_j &= -\frac{1}{2}\left(\frac{-\mathrm{sh}(\mu m\mathrm{i})\,\mathrm{ch}(\mu(2\theta_j + \mathrm{i})) + \mathrm{sh}(\mu(m\mathrm{i} - \mathrm{i}))\,\mathrm{ch}(\mu 2\theta_j)}{\mathrm{sh}(\mu \mathrm{i})} + \mathrm{ch}(\mu 2\mathrm{i}\zeta)\right)\\ &= \frac{1}{2}(\mathrm{ch}(\mu(2\theta_j + m\mathrm{i})) - \mathrm{ch}(\mu 2\mathrm{i}\zeta))\end{aligned}$$

and hence

$$k_j = \frac{x_j}{y_j} = \frac{\mathrm{sh}\big(\mu\left(\theta_j + \mathrm{i}\left(\frac{+m}{2} + \zeta\right)\right)\big)\,\mathrm{sh}\big(\mu\big(\theta_j + \mathrm{i}\left(\frac{+m}{2} - \zeta\right)\big)\big)}{\mathrm{sh}(\mu 2\theta_j)\,\mathrm{sh}(\mu \mathrm{i})}. \tag{40}$$

Specifically we take

$$x_j = x(\theta_j; m) = \mathrm{sh}\Big(\mu\Big(\theta_j + \frac{\mathrm{i}m}{2} + \mathrm{i}\zeta\Big)\Big)\,\mathrm{sh}\Big(\mu\Big(\theta_j + \frac{\mathrm{i}m}{2} - \mathrm{i}\zeta\Big)\Big) \tag{41}$$

$$y_j = z(\theta_j) = \mathrm{sh}(\mu \mathrm{i})\,\mathrm{sh}(2\mu\theta_j). \tag{42}$$

(We see that m has the role of boundary parameter.)

Note that

$$K(\theta)K(-\theta) \propto k(\theta)k(-\theta)1 + (k(\theta) + k(-\theta) + \delta_e)e = k(\theta)k(-\theta)1.$$

4. Realization via σ_l (auxiliary strings)

It remains to construct representations suitable for forming the Bethe ansatz. Our approach is to use the representations of the ordinary Temperley–Lieb algebra for which there exists a Bethe ansatz (we will concentrate on the XXZ representation) and pull them through to the blob case using the tools in section 2. (Another approach would be to generalize [46], but we do not consider that here.) As noted in section 2 we have to check that this procedure preserves the appropriate quotient inside $\mathbb{C}\mathcal{A}_{B_n}$. In *general* it does not. The first cases we consider in which it *does* are the cases of σ_l in which $l = 0, 1$. The most obvious relation obeyed by b_n cf $\mathbb{C}\mathcal{A}_{B_n}$ is (37). The representation of e will be a linear combination of that of 1 and c_0, so we require the representation of c_0 to have at most two eigenvalues. For σ_l (and general q) it is easy to check that this holds for $l = 0, 1$ only. Case $l = 0$ is the trivial solution ($K \propto 1, m = 1$), so we will focus on $l = 1$. The XXZ representation of $T_n(q)$ depends only on q, so the representation pulled through σ_1 also depends only on q, thus m must be fixed. Comparing (37), (38) and $\mathcal{R}(\sigma_1(c_0))$ we see that $m = 2$. (It may be useful to note at the outset that solutions constructed by this method turn out to coincide with known solutions. Our *new* solutions are described in section 5.)

Using the XXZ representation for $T_{n+l}(q)$ as in equation (18) we have that $\Theta : b_n \to T_{n+1} \to \mathrm{End}(V^{n+1})$ is given by $\Theta(e) = \mathcal{R}(U_1)$, $\Theta(U_i) = \mathcal{R}(U_{i+1})$. Then using (39), (42) the

K-matrix becomes

$$K(\lambda) = \begin{pmatrix} x(\lambda; 2) & & & \\ & w^-(\lambda) & z(\lambda) & \\ & z(\lambda) & w^+(\lambda) & \\ & & & x(\lambda; 2) \end{pmatrix} \tag{43}$$

with $x(\lambda; 2)$, $z(\lambda)$given by (42), and

$$w^{\pm}(\lambda) = \sinh \mu(\lambda + \mathrm{i}\zeta) \sinh \mu(\lambda - \mathrm{i}\zeta) + \mathrm{e}^{\pm 2\mu\lambda} \sinh^2(\mathrm{i}\mu). \tag{44}$$

The K-matrix can be written in the following 2×2 form:

$$\begin{aligned} K(\lambda) &= \begin{pmatrix} \underline{\alpha}(\lambda) & \underline{\beta}(\lambda) \\ \underline{\gamma}(\lambda) & \underline{\delta}(\lambda) \end{pmatrix} \\ &= \begin{pmatrix} x(\lambda; 2)1 - \frac{1}{2}\,\mathrm{e}^{\mathrm{i}\mu} z(\lambda)(1 - \sigma^z) & z(\lambda)\sigma^- \\ z(\lambda)\sigma^+ & x(\lambda; 2)1 - \frac{1}{2}\,\mathrm{e}^{-\mathrm{i}\mu} z(\lambda)(1 + \sigma^z) \end{pmatrix} \end{aligned} \tag{45}$$

where $\sigma^z, \sigma^{\pm}$ act on a two-dimensional space V_e. Note that this means that we extend the space on which the transfer matrix acts from 2^n-dimensional to 2^{n+1}. This can be considered as a system with *enhanced space* (cf [47, 48]), i.e. it is as if we added an extra site, with inhomogeneity $\mathrm{i}\zeta$, to the original spin chain. The situation is similar in quantum impurity problems (see e.g. [49–54]).

Suppose we are considering a system in which the underlying bulk model is a spin chain on V^n. Then a solution to RE is called '$\mathbb{C}$-number representation' if $K(\lambda)$ is an $N \times N$ matrix with complex entries [2, 19, 20, 55–57]. More generally, it will be evident from figures 1–3 that given any $K(\lambda)$ which satisfies RE as in (20), the 'factorized K-matrix'

$$K_f(\lambda) = R(\lambda + \mathrm{i}\zeta) K(\lambda) R(\lambda - \mathrm{i}\zeta) \tag{46}$$

where R is given by (9), is also a solution of RE. It is conjectured [58] that every solution of RE is some iteration of this construction, with a $\mathbb{C}$-number representation as base. Our solution (43) is of this factorized form with $K = 1$.

The eigenvalues of the corresponding open transfer matrix (21) can be found via the algebraic Bethe ansatz method.

4.1. The Bethe ansatz solution

Here we show explicitly how the Bethe ansatz can be applied in the case of these 'dynamical' [53] boundary conditions. (The analysis in this case is much closer to the usual setup than the 'cabled' case we will consider in section 5. We include it, since it also serves the purpose of providing a preparatory review.) We define the transfer matrix as in equation (21).

The next step is to diagonalize the transfer matrix (21) using the algebraic Bethe ansatz method. The $\mathcal{T}$-matrix (23) has the form

$$\mathcal{T}_0(\lambda) = \begin{pmatrix} A(\lambda) & B'(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} \begin{pmatrix} \underline{\alpha}(\lambda) & \underline{\beta}(\lambda) \\ \underline{\gamma}(\lambda) & \underline{\delta}(\lambda) \end{pmatrix} \begin{pmatrix} A(\lambda) & B(\lambda) \\ C'(\lambda) & D(\lambda) \end{pmatrix} = \begin{pmatrix} \mathcal{A}(\lambda) & \mathcal{B}(\lambda) \\ \mathcal{C}(\lambda) & \mathcal{D}(\lambda) \end{pmatrix} \tag{47}$$

where the matrices $\underline{\alpha}$, $\underline{\beta}$, $\underline{\gamma}$ and $\underline{\delta}$ are as in (45).

Define state $|w_+\rangle$ to be that with all spins up (the ferromagnetic vacuum vector):

$$|w_+\rangle = \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{n+1}. \tag{48}$$

Note that

$$C, C' \underbrace{\begin{pmatrix}1\\0\end{pmatrix} \otimes \cdots \otimes \begin{pmatrix}1\\0\end{pmatrix}}_{n} = 0 \qquad \underline{\gamma}(\lambda) \begin{pmatrix}1\\0\end{pmatrix} = 0 \tag{49}$$

therefore $|w_+\rangle$ is annihilated by $\mathcal{C}(\lambda)$. The operators $\mathcal{B}(\lambda)$ obey

$$[\mathcal{B}(\lambda), \mathcal{B}(\lambda')] = 0 \tag{50}$$

and act as creation operators. The Bethe state

$$|\psi\rangle = \mathcal{B}(\lambda_1) \cdots \mathcal{B}(\lambda_M)|w_+\rangle \tag{51}$$

is an eigenstate of the transfer matrix $t(\lambda)$, i.e.

$$t(\lambda)|\psi\rangle = (\mathcal{A} + \mathcal{D})|\psi\rangle = \Lambda(\lambda)|\psi\rangle. \tag{52}$$

It is easy to determine the action of $\mathcal{A}$ and $\mathcal{D}$ on the pseudo-vacuum (see below). The action of the transfer matrix on the pseudo-vacuum, cf (49), is given by

$$\begin{aligned} t(\lambda)|w_+\rangle &= \mathrm{Tr}_0 \begin{pmatrix} A(\lambda) & B'(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} \begin{pmatrix} \underline{\alpha}(\lambda) & \underline{\beta}(\lambda) \\ \underline{\gamma}(\lambda) & \underline{\delta}(\lambda) \end{pmatrix} \begin{pmatrix} A(\lambda) & B(\lambda) \\ 0 & D(\lambda) \end{pmatrix} |w_+\rangle \\ &= (\underline{\alpha}(\lambda)A^2 + \underline{\alpha}(\lambda)CB + \underline{\delta}(\lambda)D^2)|w_+\rangle. \end{aligned} \tag{53}$$

But (45) gives

$$\underline{\alpha}(\lambda) \begin{pmatrix}1\\0\end{pmatrix} = x(\lambda; 2) \begin{pmatrix}1\\0\end{pmatrix} \qquad \underline{\delta}(\lambda) \begin{pmatrix}1\\0\end{pmatrix} = w^+(\lambda) \begin{pmatrix}1\\0\end{pmatrix} \tag{54}$$

where $x(\lambda; 2)$, $w^{\pm}(\lambda)$ are given by (42), (44). We have

$$\mathcal{A}|w_+\rangle = \underline{\alpha}(\lambda)A^2|w_+\rangle \qquad \mathcal{D}|w_+\rangle = (\underline{\alpha}(\lambda)CB + \underline{\delta}(\lambda)D^2)|w_+\rangle \tag{55}$$

and finally,

$$\mathcal{A}|w_+\rangle = x(\lambda; 2)a^{2n}(\lambda)|w_+\rangle \qquad \mathcal{D}|w_+\rangle = \left(w^+(\lambda)b^{2n}(\lambda) - x(\lambda; 2)\frac{a^{2n}(\lambda) - b^{2n}(\lambda)}{a^2(\lambda) - b^2(\lambda)}\right)|w_+\rangle. \tag{56}$$

Having determined the action of the transfer matrix on the pseudo-vacuum, it is easy to see via (51), (52), (56) that knowledge of the commutation relations between $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{D}$, $\mathcal{B}$ is enough for the derivation of any eigenvalue. It is convenient [10] to consider instead of $\mathcal{D}$ the following operator:

$$\bar{\mathcal{D}} = \sinh(2\mu\lambda)\mathcal{D} - \sinh(\mathrm{i}\mu)\mathcal{A}. \tag{57}$$

Then from the fundamental relation for $\mathcal{T}$ (24) it follows that

$$\begin{aligned} \mathcal{A}(\lambda)\mathcal{B}(\lambda_i) &= X(\lambda, \lambda_i)\mathcal{B}(\lambda_i)\mathcal{A}(\lambda) + f(\lambda, \lambda_i)\mathcal{B}(\lambda)\mathcal{A}(\lambda_i) + g(\lambda, \lambda_i)\mathcal{B}(\lambda)\bar{\mathcal{D}}(\lambda_i) \\ \bar{\mathcal{D}}(\lambda)\mathcal{B}(\lambda_i) &= Y(\lambda, \lambda_i)\mathcal{B}(\lambda_i)\bar{\mathcal{D}}(\lambda) + f'(\lambda, \lambda_i)\mathcal{B}(\lambda)\mathcal{A}(\lambda_i) + g'(\lambda, \lambda_i)\mathcal{B}(\lambda)\bar{\mathcal{D}}(\lambda_i) \end{aligned} \tag{58}$$

where

$$\begin{aligned} X(\lambda, \lambda_i) &= \frac{\sinh\mu(\lambda - \lambda_i - \mathrm{i})}{\sinh\mu(\lambda - \lambda_i)} \frac{\sinh\mu(\lambda + \lambda_i - \mathrm{i})}{\sinh\mu(\lambda + \lambda_i)} \\ Y(\lambda, \lambda_i) &= \frac{\sinh\mu(\lambda - \lambda_i + \mathrm{i})}{\sinh\mu(\lambda - \lambda_i)} \frac{\sinh\mu(\lambda + \lambda_i + \mathrm{i})}{\sinh\mu(\lambda + \lambda_i)}. \end{aligned} \tag{59}$$

The other functions (f, g, f', g') are not important for our purposes since they contribute to unwanted terms, and will vanish in the final eigenvalue expression.

We can now find the eigenvalues using the above commutation relations (58), also bearing in mind (57) and the action of $\mathcal{A}$ and $\mathcal{D}$ on the pseudo-vacuum (56). The eigenvalue of any Bethe ansatz state is given by

$$\Lambda(\lambda) = \frac{\sinh\mu(\lambda+\mathrm{i}+\mathrm{i}\zeta)}{\sinh(\mu\mathrm{i})}\frac{\sinh\mu(\lambda+\mathrm{i}-\mathrm{i}\zeta)}{\sinh(\mu\mathrm{i})}\left(\frac{\sinh\mu(\lambda+\mathrm{i})}{\sinh(\mu\mathrm{i})}\right)^{2n}\frac{\sinh\mu(\lambda+\mathrm{i})}{\sinh\mu\left(\lambda+\frac{\mathrm{i}}{2}\right)}$$
$$\times\prod_{a=1}^{M}\frac{\sinh\mu\left(\lambda-\lambda_\alpha-\frac{\mathrm{i}}{2}\right)}{\sinh\mu\left(\lambda-\lambda_\alpha+\frac{\mathrm{i}}{2}\right)}\frac{\sinh\mu\left(\lambda+\lambda_\alpha-\frac{\mathrm{i}}{2}\right)}{\sinh\mu\left(\lambda+\lambda_\alpha+\frac{\mathrm{i}}{2}\right)}$$
$$+\frac{\sinh\mu(\lambda+\mathrm{i}\zeta)}{\sinh(\mu\mathrm{i})}\frac{\sinh\mu(\lambda-\mathrm{i}\zeta)}{\sinh(\mu\mathrm{i})}\left(\frac{\sinh(\mu\lambda)}{\sinh(\mu\mathrm{i})}\right)^{2n}\frac{\sinh(\mu\lambda)}{\sinh\mu\left(\lambda+\frac{\mathrm{i}}{2}\right)}$$
$$\times\prod_{\alpha=1}^{M}\frac{\sinh\mu\left(\lambda-\lambda_\alpha+\frac{3\mathrm{i}}{2}\right)}{\sinh\mu\left(\lambda-\lambda_\alpha+\frac{\mathrm{i}}{2}\right)}\frac{\sinh\mu\left(\lambda+\lambda_\alpha+\frac{3\mathrm{i}}{2}\right)}{\sinh\mu\left(\lambda+\lambda_\alpha+\frac{\mathrm{i}}{2}\right)} \tag{60}$$

provided that $\{\lambda_1,\ldots,\lambda_M\}$ are distinct and obey the Bethe ansatz equations

$$\frac{\sinh\mu\left(\lambda_\alpha+\mathrm{i}\zeta+\frac{\mathrm{i}}{2}\right)}{\sinh\mu\left(\lambda_\alpha+\mathrm{i}\zeta-\frac{\mathrm{i}}{2}\right)}\frac{\sinh\mu\left(\lambda_\alpha-\mathrm{i}\zeta+\frac{\mathrm{i}}{2}\right)}{\sinh\mu\left(\lambda_\alpha-\mathrm{i}\zeta-\frac{\mathrm{i}}{2}\right)}\left(\frac{\sinh\mu\left(\lambda_\alpha+\frac{\mathrm{i}}{2}\right)}{\sinh\mu\left(\lambda_\alpha-\frac{\mathrm{i}}{2}\right)}\right)^{2n}$$
$$=\prod_{\substack{\beta=1\\ \beta\neq\alpha}}^{M}\frac{\sinh\mu(\lambda_\alpha-\lambda_\beta+\mathrm{i})}{\sinh\mu(\lambda_\alpha-\lambda_\beta-\mathrm{i})}\frac{\sinh\mu(\lambda_\alpha+\lambda_\beta+\mathrm{i})}{\sinh\mu(\lambda_\alpha+\lambda_\beta-\mathrm{i})}\qquad \alpha=1,\ldots,M. \tag{61}$$

5. Cabling-like representation

Now consider the cabling-like representation $(\Theta : b_n \to \mathrm{End}(V^{2n}))$ from [6] discussed at the end of section 2.2. There the elements of the blob algebra are represented as follows. For $U_{n\pm l} \in T_{2n}(r)$ let $U_{n\pm l}(r) = \mathcal{R}_r(U_{n\pm l}) \in \mathrm{End}(V^{2n})$, the usual XXZ representation (18). Then

$$\Theta(U_l) = U_{\tilde{l}}(q) = U_{n-l}(r)U_{n+l}(s) \qquad \Theta(e) = U_{\tilde{0}}(Q) = \frac{1}{2\mathrm{i}\sinh(\mathrm{i}\mu)}U_n(Q) \tag{62}$$

satisfy the relations of the blob algebra $b_n(q,m)$ with

$$r = \mathrm{i}\sqrt{\mathrm{i}q} \qquad s=\sqrt{\mathrm{i}q} \qquad Q = \mathrm{i}\,\mathrm{e}^{\mathrm{i}m\mu} \tag{63}$$

(note that, $rs = -q$).

Note from (62) that the single index on a blob generator is associated with a mirror image pair in the underlying V^{2n}. The $\check{R}$-matrix is given by (19), with $\mathcal{R}(U_l) = U_{\tilde{l}}$ as defined by (62) and

$$R_{\tilde{k}\tilde{l}}(\lambda) = \sinh\mu(\lambda+\mathrm{i})\mathcal{P}_{kl}\mathcal{P}_{k'l'} + \sinh\mu\lambda\check{U}_{kl}(r)\check{U}_{k'l'}(s) \tag{64}$$

where we have introduced the space/mirror-space notation $\tilde{l} = (l,l'), \check{U}_{kl}(r) = \mathcal{P}_{kl}U_{kl}(r), \check{R}_{\tilde{k}\tilde{l}}(\lambda) = \mathcal{P}_{\tilde{k}\tilde{l}}R_{\tilde{k}\tilde{l}}(\lambda), \mathcal{P}_{\tilde{k}\tilde{l}} = \mathcal{P}_{kl}\mathcal{P}_{k'l'}$. In the R-index form here, any operator $O_{\tilde{l}} = O_{ll'}$ acts on $V_l \otimes V_{l'}$, where the $V_{l'}$ space can be considered as the 'mirror' space of V_l in the sense of figure 7(a).

It is crucial to understand the distinction between this construction and the auxiliary space construction of the previous section. There the K-matrix acts on $V_1 \otimes V_e$ with V_e a fixed auxiliary space (the space of the inserted string in figure 6—see figure 7(b)). The bulk space is essentially unchanged from that of the basic YBE solution. *Here* the entire bulk space acquires a mirror image (a mirror copy $V_{l'}$ of each V_l) and K acts on $V_1 \otimes V_{1'}$. That is, there is *no* auxiliary string.

This R-matrix satisfies the unitarity and crossing properties

$$R_{\tilde{k}\tilde{l}}(\lambda)R_{\tilde{l}\tilde{k}}(-\lambda) \propto 1 \qquad R_{\tilde{k}\tilde{l}}(\lambda) = V_{\tilde{k}} R^{t_{\tilde{l}}}_{\tilde{k}\tilde{l}}(-\lambda - \mathrm{i}) V_{\tilde{k}} \tag{65}$$

where

$$V_{\tilde{k}} = V_{kk'} = V_k(r)V_{k'}(s) \tag{66}$$

and, e.g.,

$$V_k(r) = 1 \otimes \cdots \otimes \begin{pmatrix} 0 & -\mathrm{i}r^{-\frac{1}{2}} \\ \mathrm{i}r^{\frac{1}{2}} & 0 \end{pmatrix} \otimes \cdots \otimes 1. \tag{67}$$

This R-matrix is a 16×16 matrix,

$$R(\lambda) = \begin{pmatrix} A(\lambda) & B_1(\lambda) & B_2(\lambda) & B(\lambda) \\ C_1(\lambda) & A_1(\lambda) & B_5(\lambda) & B_3(\lambda) \\ C_2(\lambda) & C_5(\lambda) & A_2(\lambda) & B_4(\lambda) \\ C(\lambda) & C_3(\lambda) & C_4(\lambda) & D(\lambda) \end{pmatrix} \tag{68}$$

where the entries shown are 4×4 matrices acting on $V \otimes V$ (see the appendix for the explicit form of the R-matrix).

The corresponding K-matrix (39), (42) is given in matrix form by the following expression (recall that $U_{\tilde{0}}$ is given by (62))

$$K(\lambda) = \begin{pmatrix} x(\lambda; m) & & & \\ & w'^{-}(\lambda) & z(\lambda) & \\ & z(\lambda) & w'^{+}(\lambda) & \\ & & & x(\lambda; m) \end{pmatrix} \tag{69}$$

where $x(\lambda; m)$, $z(\lambda)$ are given by (42) and

$$w'^{\pm}(\lambda) = x(\lambda; m) \pm \tfrac{1}{2}\,\mathrm{e}^{\mp \mathrm{i}m\mu} \sinh(2\mu\lambda). \tag{70}$$

Note that the solution here is not derived from a solution of the form in equation (46)—this is easily seen from the fact that our R-matrix is 16×16 but our K-matrix 4×4. Note in particular that the solution in the previous section, built using equations (43) and (44), is *not* simply a special case of our construction here with $m = 2$, even though the functional form is obtained in this way (consider, for example, the different structure of the underlying space).

Note, on the other hand, that the two layers in a spin ladder (as in [59, 60]) cause a doubling up of the bulk space, and thus the *bulk* space there has the same dimension as here. The spin ladder leads to a very interesting solution when considered with boundary defect [61]. However, our solution does not coincide with the solution for a spin ladder with boundary defect, as can be seen by comparing with [61]. In particular, and to reiterate, here we have a non-diagonal K-matrix *not* of form 'RKR' (i.e. *not* as in equation (46)).

The monodromy matrix has the following structure:

$$T_{\tilde{0}}(\lambda) = \begin{pmatrix} \mathcal{A}(\lambda) & \mathcal{B}_1(\lambda) & \mathcal{B}_2(\lambda) & \mathcal{B}(\lambda) \\ \mathcal{C}_1(\lambda) & \mathcal{A}_1(\lambda) & \mathcal{B}_5(\lambda) & \mathcal{B}_3(\lambda) \\ \mathcal{C}_2(\lambda) & \mathcal{C}_5(\lambda) & \mathcal{A}_2(\lambda) & \mathcal{B}_4(\lambda) \\ \mathcal{C}(\lambda) & \mathcal{C}_3(\lambda) & \mathcal{C}_4(\lambda) & \mathcal{D}(\lambda) \end{pmatrix}. \tag{71}$$

We define a reference state

$$|w_+\rangle = \underbrace{\begin{pmatrix}1\\0\end{pmatrix} \otimes \cdots \otimes \begin{pmatrix}1\\0\end{pmatrix}}_{2n} = \otimes_{i=1}^{n} |+\rangle_i \tag{72}$$

$$|+\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix}. \tag{73}$$

Then $C_i, B_5|+\rangle = 0$, i.e. $|w_+\rangle$ is annihilated by the operators $\mathcal{C}_i, \mathcal{B}_5$. Therefore, the action of the monodromy matrix on the reference state produces an upper triangular matrix,

$$T_{\tilde{0}}(\lambda)|w_+\rangle = \begin{pmatrix} \mathcal{A}(\lambda) & \mathcal{B}_1(\lambda) & \mathcal{B}_2(\lambda) & \mathcal{B}(\lambda) \\ 0 & \mathcal{A}_1(\lambda) & 0 & \mathcal{B}_3(\lambda) \\ 0 & 0 & \mathcal{A}_2(\lambda) & \mathcal{B}_4(\lambda) \\ 0 & 0 & 0 & \mathcal{D}(\lambda) \end{pmatrix} |w_+\rangle. \tag{74}$$

Thus for the bulk case (71), (7) the pseudo-vacuum eigenvalue is given by

$$t(\lambda)|w_+\rangle = (\mathcal{A} + \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{D})|w_+\rangle = (a^n(\lambda) + b^n(\lambda))|w_+\rangle \tag{75}$$

where

$$\mathcal{A}(\lambda) = \prod_{l=1}^{n} A^{\tilde{l}} \qquad \mathcal{A}_1(\lambda) = \prod_{l=1}^{n} A_1^{\tilde{l}} \qquad \mathcal{A}_2(\lambda) = \prod_{l=1}^{n} A_2^{\tilde{l}} \qquad \mathcal{D}(\lambda) = \prod_{l=1}^{n} D^{\tilde{l}} \tag{76}$$

(see also the appendix).

Now consider the open transfer matrix (21),

$$t(\lambda) = \mathrm{Tr}_{\tilde{0}}\, M_{\tilde{0}} T_{\tilde{0}}(\lambda) K_{\tilde{0}}(\lambda) T_{\tilde{0}}^{-1}(-\lambda) \tag{77}$$

where $K_{\tilde{0}} = K_{00'}$ given by (69) and

$$M_{\tilde{0}} = V_{\tilde{0}}^{t} V_{\tilde{0}}. \tag{78}$$

Then the pseudo-vacuum eigenvalue will be

$$\begin{aligned}\Lambda^0(\lambda) = \langle w_+|(&q x(\lambda; m)\mathcal{A}^2 + q^{-1}x(\lambda; m)\mathcal{D}^2 + q^{-1}x(\lambda; m)\mathcal{C}\mathcal{B} + \mathrm{i}x(\lambda; m)\mathcal{C}_1\mathcal{B}_1 \\ &- \mathrm{i}x(\lambda; m)\mathcal{C}_2\mathcal{B}_2 + q^{-1}w'^{-}(\lambda)\mathcal{C}_3\mathcal{B}_3 + q^{-1}w'^{+}(\lambda)\mathcal{C}_4\mathcal{B}_4)|w_+\rangle.\end{aligned} \tag{79}$$

where $\mathcal{A}, \mathcal{D}$ are given by (76) and

$$\begin{aligned} \mathcal{C}_{1,2}(\lambda) &= \prod_{l=1}^{n-1} A^{\tilde{l}} C_{1,2}^{\tilde{n}} \qquad & \mathcal{B}_{1,2}(\lambda) &= \prod_{l=1}^{n-1} A^{\tilde{l}} B_{1,2}^{\tilde{n}} \\ \mathcal{C}_{3,4}(\lambda) &= \prod_{l=2}^{n} D^{\tilde{l}} C_{3,4}^{\tilde{1}} \qquad & \mathcal{B}_{3,4}(\lambda) &= \prod_{l=2}^{n} D^{\tilde{l}} B_{3,4}^{\tilde{1}} \end{aligned} \tag{80}$$

and

$$\begin{aligned}\mathcal{C}(\lambda) = &\sum_{l=1}^{n} D^{\tilde{n}} \cdots D^{\widetilde{l+1}} C^{\tilde{l}} A^{\widetilde{l-1}} \cdots A^{\tilde{1}} + \sum_{l=1}^{n-1} D^{\tilde{n}} \cdots D^{\widetilde{l+2}} C_4^{\widetilde{l+1}} C_2^{\tilde{l}} A^{\widetilde{l-1}} \cdots A^{\tilde{1}} \\ &+ \sum_{l=1}^{n-1} D^{\tilde{n}} \cdots D^{\widetilde{l+2}} C_3^{\widetilde{l+1}} C_1^{\tilde{l}} A^{\widetilde{l-1}} \cdots A^{\tilde{1}}\end{aligned} \tag{81}$$

$$\begin{aligned}\mathcal{B}(\lambda) = &\sum_{l=1}^{n} D^{\tilde{n}} \cdots D^{\widetilde{l+1}} B^{\tilde{l}} A^{\widetilde{l-1}} \cdots A^{\tilde{1}} + \sum_{l=1}^{n-1} D^{\tilde{n}} \cdots D^{\widetilde{l+2}} B_4^{\widetilde{l+1}} B_2^{\tilde{l}} A^{\widetilde{l-1}} \cdots A^{\tilde{1}} \\ &+ \sum_{l=1}^{n-1} D^{\tilde{n}} \cdots D^{\widetilde{l+2}} B_3^{\widetilde{l+1}} B_1^{\tilde{l}} A^{\widetilde{l-1}} \cdots A^{\tilde{1}}.\end{aligned} \tag{82}$$

It is also useful to derive the action of the following operators on the $|+\rangle$ state:

$$
\begin{aligned}
&A^2|+\rangle = a^2(\lambda)|+\rangle \qquad D^2|+\rangle = b^2(\lambda)|+\rangle \\
&C_1B_1|+\rangle = a^2(\lambda)|+\rangle \qquad C_2B_2|+\rangle = a^2(\lambda)|+\rangle \\
&CB|+\rangle = (a(\lambda) - qb(\lambda))(a(\lambda) - q^{-1}b(\lambda))|+\rangle \\
&C_3B_3|+\rangle = b^2(\lambda)|+\rangle \qquad C_4B_4|+\rangle = b^2(\lambda)|+\rangle.
\end{aligned}
\tag{83}
$$

Taking into account equations (79)–(83) we conclude that the pseudo-vacuum eigenvalue has the form

$$\Lambda^0(\lambda) = f_1(\lambda)a(\lambda)^{2n} + f_2(\lambda)b(\lambda)^{2n} \tag{84}$$

where the functions $f_1(\lambda)$, $f_2(\lambda)$ are due to the boundary, and are determined explicitly by (79)–(83).[4] The important observation here is that we are able to derive the pseudo-vacuum eigenvalue explicitly. Furthermore, we note that it has the expected form, compared to the corresponding bulk eigenvalue (75), in as much as the powers of a and b are doubled in the open chain, and the functions f_1 and f_2 appear as a result of the presence of the boundary. The next step is the derivation of the general Bethe ansatz state and the corresponding eigenvalue. Here, we do not give the details of this derivation. However, we conjecture that the general eigenvalue will have the following form

$$\Lambda(\lambda) = f_1(\lambda)a(\lambda)^{2n}\mathfrak{A}_1(\lambda) + f_2(\lambda)b(\lambda)^{2n}\mathfrak{A}_2(\lambda) \tag{85}$$

where $\mathfrak{A}_1(\lambda), \mathfrak{A}_2(\lambda)$ can be determined explicitly via the algebraic or the analytical Bethe ansatz method. We will report on the detailed analysis of the Bethe ansatz eigenstates and eigenvalues, which is a separate interesting problem, in a future work.

We have arrived at this solution from abstract considerations, however, it clearly describes a spin-chain model and it does not coincide with any known solution. Furthermore, we have retained complete freedom of choice of the boundary parameter m. This model also has interesting symmetry properties which appear to significantly generalize the role of $U_q sl_2$ for ordinary XXZ: this makes the model a very interesting candidate for study, and a full spectrum analysis. From the representation theory of b_n [13] we know that $T_n(q)$ appears in b_n in two different ways—as a subalgebra on dropping the boundary generator e, and as a quotient for the special boundary parameter choice $m = 1$. We also know that the structure of b_n depends profoundly on the boundary parameter m. It will be interesting to see how the spectrum of $t(\lambda)$ depends on m, and also how the connections with $T_n(q)$ relate the spectrum of $t(\lambda)$ here to that in the ordinary XXZ case. Indeed it is an interesting (and hopefully simpler) preliminary question to ask what is the spectrum of $t(\lambda)$ in this 'representation' *without* the boundary term. (For example, does this spectrum still depend on r and s separately?) This should give an insight into the spectrum *with* boundary.

6. The Hamiltonian

Here we derive the Hamiltonians of the auxiliary string and cabling realizations.

[4] $f_1(\lambda) = \dfrac{x(\lambda)}{a^2(\lambda) - b^2(\lambda)}\{q(a^2(\lambda) - b^2(\lambda)) + q^{-1}(a^2(\lambda) + 3b^2(\lambda) - (q + q^{-1})a(\lambda)b(\lambda))\}$

$f_2(\lambda) = \dfrac{q^{-1}}{a^2(\lambda) - b^2(\lambda)}\{(x(\lambda) + w'^{+}(\lambda) + w'^{-}(\lambda))(a^2(\lambda) - b^2(\lambda)) - x(\lambda)(3a^2(\lambda) + b^2(\lambda) - (q + q^{-1})a(\lambda)b(\lambda))\}.$

6.1. The auxiliary string realization

The open spin-chain Hamiltonian $\mathcal{H}$ is related to the derivative of the transfer matrix at $\lambda = 0$:

$$\mathcal{H} = \sum_{m=1}^{n-1} \mathcal{H}_{mm+1} + \frac{1}{4\mu x(\lambda; 2)} \frac{\mathrm{d}}{\mathrm{d}\lambda} K_1(\lambda)\bigg|_{\lambda=0} + \frac{\mathrm{Tr}_0 M_0 \mathcal{H}_{n0}}{\mathrm{Tr}\, M} \tag{86}$$

where $x(\lambda; m)$ is given by (42), and the two-site Hamiltonian $\mathcal{H}_{jk}$ is given by

$$\mathcal{H}_{jk} = \frac{1}{2\mu} \mathcal{P}_{jk} \frac{\mathrm{d}}{\mathrm{d}\lambda} R_{jk}(\lambda)\bigg|_{\lambda=0} \tag{87}$$

where the R-matrix is given by (19). This Hamiltonian is Hermitian.

Consider the model defined by the Hamiltonian in (87), (86):

$$\begin{aligned}\mathcal{H} = \frac{1}{4} \sum_{i=1}^{n-1} &\left(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \cosh(\mathrm{i}\mu)\sigma_i^z \sigma_{i+1}^z\right) + \frac{\sinh(\mathrm{i}\mu)}{4}\left(\sigma_n^z - \sigma_1^z\right) \\ &+ \frac{\sinh(\mathrm{i}\mu)}{4x(0; 2)}\left(\sigma_e^x \sigma_1^x + \sigma_e^y \sigma_1^y + \cosh(\mathrm{i}\mu)\sigma_e^z \sigma_1^z\right) + \frac{\sinh^2(\mathrm{i}\mu)}{4x(0; 2)}\left(\sigma_e^z - \sigma_1^z\right) \\ &+ n\cosh(\mu \mathrm{i})/4\end{aligned} \tag{88}$$

where σ_e^i act on the extra space of the chain. The bulk part is the usual XXZ bulk spin chain with first neighbour interaction. The last two terms describe the boundary interaction and come from the derivative of the K-matrix. This Hamiltonian describes a model which is coupled to a quantum mechanical (spin) system at the boundaries. Note that it is nothing more than an $(n + 1)$-site Hamiltonian with an inhomogeneity at the end.

Consider the Hamiltonian $\mathcal{H}_f$ obtained when we take boundaries of the form (46), where K is the diagonal matrix [19, 20, 57]

$$K(\lambda) = \mathrm{diag}(\sinh \mu(-\lambda + \mathrm{i}\xi)\, \mathrm{e}^{\mu\lambda}, \sinh \mu(\lambda + \mathrm{i}\xi)\, \mathrm{e}^{-\mu\lambda}). \tag{89}$$

By direct computation we find here

$$\begin{aligned}\mathcal{H}_f = \mathcal{H} + &\frac{\coth(\mathrm{i}\mu\xi) - 1}{4x(0; 2)}\left(\sinh^2(\mathrm{i}\mu\zeta)\sigma_e^z - \sinh^2(\mathrm{i}\mu)\sigma_1^z\right) \\ &+ (\coth(\mathrm{i}\mu\xi) - 1)\frac{\sinh(\mathrm{i}\mu)\sinh(\mathrm{i}\mu\zeta)}{2x(0; 2)} F_e(\zeta) G_1(-\zeta) + n\cosh(\mu \mathrm{i})/4\end{aligned} \tag{90}$$

where $\mathcal{H}$ is from (88) and

$$F(\zeta) = \begin{pmatrix} 0 & \mathrm{e}^{\mathrm{i}\mu\frac{\zeta}{2}} \\ \mathrm{e}^{-\mathrm{i}\mu\frac{\zeta}{2}} & 0 \end{pmatrix} \qquad G(\zeta) = \begin{pmatrix} 0 & \mathrm{e}^{\mathrm{i}\mu\frac{\zeta}{2}} \\ -\mathrm{e}^{-\mathrm{i}\mu\frac{\zeta}{2}} & 0 \end{pmatrix}. \tag{91}$$

For $\zeta = 0$ the above matrices become proportional to σ^x and σ^y, respectively. For $\mathrm{i}\xi \to \infty$ we see that $\mathcal{H}_f$ coincides with $\mathcal{H}$. Interestingly, the Hamiltonian $\mathcal{H}_f$ does not appear to have been written down explicitly before.

6.2. The cabling representation

Note from (64) that for $\lambda = 0$ the R-matrix reduces to a product of two permutation operators. Therefore, the corresponding local Hamiltonian is defined:

$$\mathcal{H}_{\mathrm{open}} = \sum_{l=1}^{n-1} \mathcal{H}_{\tilde{l}\,\widetilde{l+1}} + \frac{1}{4\mu x(\lambda; m)} \frac{\mathrm{d}}{\mathrm{d}\lambda} K_{\tilde{1}}(\lambda)\bigg|_{\lambda=0} + \frac{\mathrm{Tr}_{\tilde{0}} M_{\tilde{0}} \mathcal{H}_{\tilde{n}\tilde{0}}}{\mathrm{Tr}\, M} \tag{92}$$

where the two-site Hamiltonian $\mathcal{H}_{\tilde k\tilde l}$ is given by

$$\mathcal{H}_{\tilde k\tilde l} = \frac{1}{2\mu}\mathcal{P}_{\tilde k\tilde l}\,\frac{\mathrm{d}}{\mathrm{d}\lambda}R_{\tilde k\tilde l}(\lambda)\Big|_{\lambda=0} \tag{93}$$

and the R-matrix is given by (64). Unlike (88) $\mathcal{H}_{\mathrm{open}}$ is completely new. It has some structural similarity with the case in [61] for the bulk part $\mathcal{H}_{\tilde k\tilde l}$, in the sense that they operate locally on the same space, so let us explicitly contrast with this case. Our $\mathcal{H}_{\mathrm{open}}$ may be written explicitly as follows. Let

$$\hat{Q}_{\tilde k\tilde l} = \frac{-1}{\sinh(\mu\mathrm{i})}\mathcal{P}_{\tilde k\tilde l}R_{\tilde k\tilde l}(-\mathrm{i})$$

$$= \begin{pmatrix}
0 & & & & & & & & & & & & & & & \\
 & 0 & & & & & & & & & & & & & & \\
 & & 0 & & & & & & & & & & & & & \\
 & & & -q & & & -r & & & -s & & & 1 & & & \\
 & & & & 0 & & & & & & & & & & & \\
 & & & & & 0 & & & & & & & & & & \\
 & & & -r & & & \mathrm{i} & & & 1 & & & -s^{-1} & & & \\
 & & & & & & & 0 & & & & & & & & \\
 & & & & & & & & 0 & & & & & & & \\
 & & & -s & & & 1 & & & -\mathrm{i} & & & -r^{-1} & & & \\
 & & & & & & & & & & 0 & & & & & \\
 & & & & & & & & & & & 0 & & & & \\
 & & & 1 & & & -s^{-1} & & & -r^{-1} & & & -q^{-1} & & & \\
 & & & & & & & & & & & & & 0 & & \\
 & & & & & & & & & & & & & & 0 & \\
 & & & & & & & & & & & & & & & 0
\end{pmatrix}. \tag{94}$$

(here $R(\lambda)$ is as given in the appendix). Then

$$\mathcal{H}_{\mathrm{open}} = \frac{1}{2}\sum_{l=1}^{n-1}\hat{Q}_{\tilde l\,\widetilde{l+1}} + n\frac{\cosh(\mu\mathrm{i})}{2} + \frac{1}{4x(0;m)}\left(D_{\tilde 1}^{(-)} + \sinh(\mu\mathrm{i})\mathcal{P}_{\tilde 1}\right) - \frac{1}{2\cosh(\mu\mathrm{i})}D_{\tilde n}^{(+)} \tag{95}$$

where

$D^{(-)} = \mathrm{Diagonal}(\sinh(\mathrm{i}\mu m) - \sinh(\mathrm{i}\mu), -\cosh(\mathrm{i}\mu m), \cosh(\mathrm{i}\mu m), \sinh(\mathrm{i}\mu m) - \sinh(\mathrm{i}\mu))$

$D^{(+)} = \mathrm{Diagonal}(q^2, -1, -1, q^{-2})$, and $\mathcal{P}_{\tilde 1}$ is the 4×4 permutation operator acting on $V_1\otimes V_{1'}$. On the other hand, Wang and Schlottmann's model I is

$$H_{\mathrm{I}} = \sum_{l=1}^{n-1}\mathcal{P}_{\tilde l\,\widetilde{l+1}} - J\sum_{l=2}^{n}X_{\tilde l}^{00} + (n-1)J/4 - J'X_{\tilde 1}^{00} + \frac{1}{4}J' \tag{96}$$

and their model II is

$$H_{\mathrm{II}} = \sum_{l=2}^{n-1}\mathcal{P}_{\tilde l\,\widetilde{l+1}} - J\sum_{l=1}^{n}X_{\tilde l}^{00} + nJ/4 + U\mathcal{P}_{\tilde 1\tilde 2} \tag{97}$$

where $\mathcal{P}_{\tilde l\,\widetilde{l+1}}$ is again the 16×16 permutation operator acting on $V_l\otimes V_{l'}\otimes V_{l+1}\otimes V_{(l+1)'} = V_{\tilde l}\otimes V_{\widetilde{l+1}}$ (here V_l and $V_{l'}$ are the two spaces at either end of the Jth rung of the ladder); and $X_{\tilde l}^{00} = -\frac{1}{2}\mathcal{P}_{\tilde l} + \frac{1}{2}$; and J and U, J' are the bulk and boundary coupling constants, respectively.

Comparing $\hat{Q}$ (as in equation (94)) with $\mathcal{P}$ one finds that these matrices are alike *only* in so far as they are both 16×16. There is no choice of the parameter μ such that $\hat{Q}$ specializes to $\mathcal{P}$. Similarly there is no choice of J, for any μ, to make the overall bulk terms coincide. The most striking difference, perhaps, is that Wang and Schlottmann's bulk model is invariant under the usual $SU(4)$ action on tensor space when $J = 0$, while ours is never so! The symmetry properties of our model remain an interesting open question.

Finally, we write explicitly the isotropic case of our model. The isotropic case requires a little subtlety to construct without divergences: $\delta_e = -2\cosh(y\mathrm{i})$, $\kappa = \cosh(y\mathrm{i})$. We have

$$\mathcal{H}_{\text{open/isotropic}} = \frac{1}{2}\sum_{l=1}^{n-1} \hat{Q}_{\widetilde{\tilde{l}\,l+1}}(\mu = 0) + \frac{n}{2} + \frac{1}{2c}\left(E_{\tilde{1}}^{(-)} + \mathcal{P}_{\tilde{1}}\right) - \frac{1}{2}D_{\tilde{n}}^{(+)}(\mu = 0) \tag{98}$$

where c is an arbitrary constant,

$$E^{(-)} = \text{Diagonal}(\cosh(y\mathrm{i}) - 1, \sinh(y\mathrm{i}), -\sinh(y\mathrm{i}), \cosh(y\mathrm{i}) - 1)$$

and y is a boundary parameter.

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Appendix

In this section we write explicitly the 16×16 $R_{\tilde{k}\tilde{l}}$ matrix. In particular, we write down the 4×4 entries of the matrix,

$$A(\lambda) = \begin{pmatrix} a(\lambda) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b(\lambda) \end{pmatrix} \qquad D(\lambda) = \begin{pmatrix} b(\lambda) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a(\lambda) \end{pmatrix} \tag{99}$$

$$A_1(\lambda) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a(\lambda) & 0 & 0 \\ 0 & 0 & b(\lambda) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad A_2(\lambda) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & b(\lambda) & 0 & 0 \\ 0 & 0 & a(\lambda) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{100}$$

$$B_1(\lambda) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ a(\lambda) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -s^{-1}b(\lambda) & 0 \end{pmatrix} \qquad B_2(\lambda) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a(\lambda) & 0 & 0 & 0 \\ 0 & -r^{-1}b(\lambda) & 0 & 0 \end{pmatrix} \tag{101}$$

$$B_5(\lambda) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & a(\lambda) - r^{-1}sb(\lambda) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad B(\lambda) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a(\lambda) - q^{-1}b(\lambda) & 0 & 0 & 0 \end{pmatrix} \tag{102}$$

B_3, B_4 have the same structure as B_2, B_1 respectively, with the matrix elements interchanged. Also, $C_i(p) = B_i(p^{-1})^t$, where p is in general the anisotropy parameter; it can be *r, s, q*.

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